An elementary proof of the robustness of the linear hyperbolic flows

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Abstract

We present an elementary proof that the qualitative picture of a linear hyperbolic flow is insensitive to slight measurements errors in its associated vector field.

1 Robustness of the linear hyperbolic flows

We start considering linear flows e^{tH}, where the matrix H is hyperbolic, i.e., all its eigenvalues have nonzero real part. The set $\mathcal{H}$ of all hyperbolic matrices is generic, in the sense that it is an open and dense subset of the set $\mathcal{M}$ of all d by d matrices. It is not difficult to see the density of $\mathcal{H}$. In fact, λ is an eigenvalue of an arbitrary matrix A if and only if $\lambda + \varepsilon$ is an eigenvalue of the matrix $A + \varepsilon I$. Hence, for any positive ε lower than the modulus of the real part of each eigenvalue of A with nonzero real part, the matrix $A + \varepsilon I$ is hyperbolic and thus can be set arbitrarily closed to A. On the other hand, the proof of the openness of $\mathcal{H}$ is not so immediate, since it involves some kind of continuity of the real part of the eigenvalues.

In this note, we provide an elementary proof of the following stronger fact: the set $\mathcal{H}_s$ of all hyperbolic matrices having exactly s eigenvalues with negative real part is open in $\mathcal{M}$. Since $\mathcal{H}$ is partitioned in these subsets, its openness is thus immediate. This partition is related to the qualitative classification of the phase portraits of the linear hyperbolic flows. It is well

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known (see, e.g., [1] and [2]) that two linear hyperbolic flows e^{tH_1} and e^{tH_2} have equivalent phase portraits (are topologically conjugated) if and only if H_1 and H_2 belong to the same subset $\mathcal{H}_s$. Thus the openness of these subsets implies that the qualitative picture of a linear hyperbolic flow e^{tH} is insensitive to slight measurements errors in H, what is called *the robustness of the linear hyperbolic flows.*

Theorem 1.1 *The linear hyperbolic flows are robust.*

Proof: First we introduce the following notation. For a given matrix A, we denote $s(A)$ the number of eigenvalues of A (counting multiplicity) with negative real part and $u(A)$ the number of eigenvalues of A (counting multiplicity) with positive real part. The integers $s(A)$ and $u(A)$ are called, respectively, the stable and the unstable dimensions of A. For a hyperbolic matrix H we have that $H \in \mathcal{H}_{s(H)}$ and that $s(H) + u(H) = d$.

In order to show that $\mathcal{H}_s$ is open, it is sufficient to show that, for all $H \in \mathcal{H}_s$ and all sequence $A_n \to H$, we have that $s(A_n) \to s(H)$ and that $u(A_n) \to u(H)$. In fact, if $\mathcal{H}_s$ is not open, there exist $H \in \mathcal{H}_s$ and $A_n \to H$ such that $A_n \notin \mathcal{H}_s$, for all $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$, we have that $s(A_n) \neq s(H)$ or $u(A_n) \neq u(H)$, which contradicts $s(A_n) \to s(H)$ and $u(A_n) \to u(H)$.

Now we fix $H \in \mathcal{H}_s$ and a sequence $A_n \to H$. Considering the continuous function $p : \mathcal{M} \times \mathbb{C} \to \mathbb{C}$ given by $p(A, z) = \det(A - zI)$, we have that $p(A, z)$ is the characteristic polynomial of A. We denote $p(A_n, z)$ and $p(H, z)$, respectively, by $p_n(z)$ and $p(z)$. If $\lambda_n \to \lambda$ and λ_n is an eigenvalue of A_n, then λ is an eigenvalue of H, since $0 = p_n(\lambda_n) \to p(\lambda)$, which shows that $p(\lambda) = 0$. Denoting by $\{\lambda_n^1, \ldots, \lambda_n^d\}$ the eigenvalues of A_n and by $\{\lambda^1, \ldots, \lambda^d\}$ the eigenvalues of H, we have that

$$p_n(z) = (z - \lambda_n^1) \cdots (z - \lambda_n^d) \qquad \text{and} \qquad p(z) = (z - \lambda^1) \cdots (z - \lambda^d).$$

On the other hand, we have that

$$\lambda_n^1 \cdots \lambda_n^d = \det A_n \to \det H = \lambda^1 \ldots \lambda^d \neq 0,$$

which implies that $(\lambda_n^1, \ldots, \lambda_n^d)$ is bounded sequence in $\mathbb{C}^d$. In fact, if not, there should exist a subsequence $\lambda_{n_k}^i \to \infty$, for some $1 \leq i \leq d$. Since the product $\lambda_n^1 \cdots \lambda_n^d$ is bounded, there should exist a sequence $\lambda_{n_k} \to 0$, where λ_{n_k} is an eigenvalue of A_{n_k}. In this case, since $A_{n_k} \to H$, we get that 0 is an eigenvalue of H, which is not possible.

Now fix an arbitrary subsequence A_{n_k}. The Bolzano-Weierstrass theorem implies that, for each $1 \leq i \leq d$, there exist $\alpha^i \in \mathbb{C}$ and a sub-subsequence $\lambda^i_{n_{k_l}}$ such that $\lambda^i_{n_{k_l}} \to \alpha^i$. Then for each $z \in \mathbb{C}$, we have that

$$p_{n_{k_l}}(z) = (z - \lambda^1_{n_{k_l}}) \cdots (z - \lambda^d_{n_{k_l}}) \to (z - \alpha^1) \cdots (z - \alpha^d).$$

On the other hand, since $A_{n_{k_l}} \to H$, we have that

$$p_{n_{k_l}}(z) \to p(z) = (z - \lambda^1) \cdots (z - \lambda^d),$$

showing that

$$(z - \alpha^1) \cdots (z - \alpha^d) = (z - \lambda^1) \cdots (z - \lambda^d),$$

for each $z \in \mathbb{C}$. This implies that there exists a permutation σ of the set $\{1, \ldots, d\}$ such that $\alpha^i = \lambda^{\sigma(i)}$ and thus that $\lambda^i_{n_{k_l}} \to \lambda^{\sigma(i)}$, for each $1 \leq i \leq d$. Hence there exists $l_0 \in \mathbb{N}$ such that the signal of the real part of $\lambda^i_{n_{k_l}}$ and $\lambda^{\sigma(i)}$ coincides for all $l \geq l_0$ and all $1 \leq i \leq d$. This implies that $s(A_{n_{k_l}}) = s(H)$ and $u(A_{n_{k_l}}) = u(H)$, for all $l \geq l_0$.

Hence, for every arbitrary subsequence A_{n_k}, there exists a sub-subsequence such that $s(A_{n_{k_l}}) \to s(H)$ and $u(A_{n_{k_l}}) \to u(H)$. This already implies that $s(A_n) \to s(H)$ and $u(A_n) \to u(H)$, completing our proof. $\square$

References

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[2] M. C. Irwin: *Smooth dynamical systems*, Academic Press, New York (1980).